

Ergodic Theory - Week 10

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1 Spectral theory of measure-preserving systems

P1. Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and let $f \in L^2(X)$. Let μ_f be the spectral measure of f and let $P_T f$ be the orthogonal projection onto the closed subspace of T -invariant functions in $L^2(X)$. Show that:

(a) Show that for every $t \in \mathbb{T}$, we have

$$\mu_f(\{t\}) = \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} e(-nt) \int \bar{f} \cdot T^n f d\mu.$$

Conclude that $\mu_f(\{0\}) = \|P_T f\|_2^2$, where P_T denotes the projection to the space of invariant functions.

(b) If the system is ergodic, then show that $\mu_f(\{0\}) = 0$ if and only if $\int f d\mu = 0$.

(c) $\mu_f(\mathbb{T}) = \|f\|_2^2$.

(a) Using the spectral theorem, we deduce that there exists a measure on $\mathbb{T}$ such that

$$\int \bar{f} \cdot T^n f d\mu = \int e(nx) d\mu_f(x)$$

for every $n \in \mathbb{N}$. Therefore, the average in the right-hand side of our inequality is equal to

$$\frac{1}{N} \sum_{n=0}^{N-1} e(-nt) \int e(nx) d\mu_f(x) = \frac{1}{N} \sum_{n=0}^{N-1} \int e(n(x-t)) d\mu_f(x) = \int \frac{1}{N} \sum_{n=0}^{N-1} e(n(x-t)) d\mu_f(x).$$

Observe that

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} e(n(x-t)) = \begin{cases} 1, & \text{if } x = t \\ 0, & \text{otherwise} \end{cases}.$$

Using the dominated convergence theorem, we deduce that

$$\begin{aligned} \lim_{N \rightarrow +\infty} \int \frac{1}{N} \sum_{n=0}^{N-1} e(n(x-t)) d\mu_f(x) &= \int \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} e(n(x-t)) d\mu_f(x) = \\ &= \int \mathbf{1}_{x=t} d\mu_f(x) = \mu_f(\{t\}). \end{aligned}$$

In the case $t = 0$, we have

$$\mu_f(\{0\}) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \int \bar{f} \cdot T^n f = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle T^n f, f \rangle.$$

Now, write $f = P_T f + f_{\text{erg}}$ where $P_T f$ is the invariant part of f and f_{erg} is the coboundary part of f . Using the mean ergodic theorem and the fact that $P_T f$ and f_{erg} are orthogonal, we get that

$$\begin{aligned} \mu_f(\{0\}) &= \lim_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{n=0}^{N-1} \langle T^n P_T f, P_T f \rangle + \langle T^n f_{\text{erg}}, P_T f \rangle + \langle T^n P_T f, f_{\text{erg}} \rangle + \langle T^n f_{\text{erg}}, f_{\text{erg}} \rangle \right) \\ &= \lim_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{n=0}^{N-1} \langle P_T f, P_T f \rangle + \langle T^n f_{\text{erg}}, f_{\text{erg}} \rangle \right) \\ &= \langle P_T f, P_T f \rangle = \|P_T f\|_2^2, \end{aligned}$$

where in the last line we used the fact that the ergodic averages of f_{erg} converge to zero in $L^2(X)$.

(b) If T is ergodic, then $P_T f = \int f d\mu$. Thus

$$\mu_f(\{0\}) = \left\| \int f d\mu \right\|_2^2 = \left(\int f d\mu \right)^2,$$

from which the equivalence follows.

(c) Using the spectral theorem we have that

$$\mu(\mathbb{T}) = \int_{\mathbb{T}} d\mu_f = \int_X f \cdot \bar{f} d\mu = \|f\|_2^2.$$

P2. Let $\{a(n)\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{N}$ such that for every $\beta \in (0, 1)$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e(a(n)\beta) = 0.$$

Suppose (X, μ, T) is an ergodic measure-preserving system. Let $f \in L^2(\mu)$. Prove that

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N T^{a(n)} f - P_T f \right\|_{L^2(\mu)} = 0,$$

where P_T denotes the projection to the space of invariant functions.

Use this exercise to give another proof of von Neumann's mean ergodic theorem.

Solution: Let $\beta \in (0, 1)$ and $f \in L^2(\mu)$. We can write $f = g + P_T f$ and the problem reduces to

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N T^{a(n)} g \right\|_{L^2(\mu)} = 0.$$

Let μ_g be the spectral measure associated to g . Then

$$\begin{aligned}
\left\| \frac{1}{N} \sum_{n=1}^N T^{a(n)} g \right\|_{L^2(\mu)}^2 &= \int \left| \frac{1}{N} \sum_{n=1}^N T^{a(n)} g \right|^2 d\mu \\
&= \frac{1}{N^2} \int \sum_{n,k=1}^N T^{a(n)} g \cdot \overline{T^{a(k)} g} d\mu \\
&= \frac{1}{N^2} \sum_{n=1}^N \sum_{k=1}^N \langle T^{a(n)} g, T^{a(k)} g \rangle \\
&= \frac{1}{N^2} \sum_{n=1}^N \sum_{k=1}^N \langle T^{a(n)-a(k)} g, g \rangle \\
&= \frac{1}{N^2} \sum_{n=1}^N \sum_{k=1}^N \int_{\mathbb{T}} e((a(n) - a(k))x) d\mu_g(x) \\
&= \int_{\mathbb{T}} \left(\frac{1}{N} \sum_{n=1}^N e(a(n)x) \right) \left(\frac{1}{N} \sum_{k=1}^N e(-a(k)x) \right) d\mu_g(x) \\
&= \int_{\mathbb{T}} \left| \frac{1}{N} \sum_{n=1}^N e(a(n)x) \right|^2 d\mu_g(x).
\end{aligned}$$

As $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e(a(n)x) \rightarrow 0$ for every $x \in \mathbb{T}$ except zero (where the limit of the average is 1), by the dominated convergence theorem, we have that the last bound tends to $\mu_g(\{0\})$ as $N \rightarrow \infty$. However, $\mu_g(\{0\}) = P_T g = 0$ by Exercise 1a). Therefore,

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N T^{a(n)} f - P_T f \right\|_{L^2(\mu)} = 0.$$

P3. Let ν be any continuous (that is $\nu(\{x\}) = 0$ for all $x \in \mathbb{T}$) Borel probability measure on the torus $\mathbb{T}$ that is invariant under the maps $T_p x = px \pmod{1}$ for all $p \in \mathbb{N}$. Show that ν is the Lebesgue measure (a famous conjecture of Furstenberg asserts that the same conclusion holds under the much weaker assumption that the measure is invariant under T_2 and T_3).

Hint: For a finite Borel measure μ on $\mathbb{T}$, we define its Fourier coefficients by $\widehat{\mu}(n) = \int e(nt) d\mu(t)$. Use the fact that if two measures have the same Fourier coefficients, then they are equal.

We will show that the measures ν and λ (the Lebesgue measure on $\mathbb{T}$) have the same Fourier coefficients, that is $\widehat{\nu}(n) = \widehat{\lambda}(n)$. If we show this, then we use linearity to deduce that $\int_{\mathbb{T}} P(x) d\nu(x) = \int_{\mathbb{T}} P(x) d\lambda(x)$ for any trigonometric polynomial and then by density we get that for any function in $f \in \mathbb{C}(\mathbb{T})$ we have $\int_{\mathbb{T}} f(x) d\nu(x) = \int_{\mathbb{T}} f(x) d\lambda(x)$. Using the regularity of the measures, we can replace any $f \in \mathbb{C}(\mathbb{T})$ with indicator functions of a set, and we are done.

We easily compute

$$\widehat{\lambda}(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

For all $n \in \mathbb{N} \setminus \{0\}$, we have that

$$\widehat{v}(n) = \int_0^1 e^{2\pi i n t} dv(t) = \int_0^1 e^{2\pi i (T_n t)} dv(t) = \int_0^1 e^{2\pi i t} dv(t) = \widehat{v}(1),$$

where we used in the third equality the fact that the measure ν is T_n invariant.

We also easily see that $\widehat{v}(-n) = \overline{\widehat{v}(n)} = \overline{\widehat{v}(1)}$ (the first equality holds for all measures).

To finish the proof, we use Wiener's lemma and the fact that the measure is non-atomic to get that

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\widehat{v}(n)|^2 = 0.$$

Since $|\widehat{v}(n)|^2 = |\widehat{v}(1)|^2$, we conclude that $\widehat{v}(n) = 0$ for all $n \geq 1$.

Putting everything together, we get that $\widehat{v}(n) = \widehat{\lambda}(n)$ for all $n \in \mathbb{Z}$ and, thus, $v = \lambda$.

P4. In this exercise, we outline the steps to prove Sárközy's theorem: if $(X, \mathcal{B}, \mu, T)$ is a measure-preserving system and $A \in \mathcal{B}$ has positive measure, then there exist infinitely many $n \in \mathbb{N}$ such that $\mu(A \cap T^{-n^2} A) > 0$.

This can be combined with a well-known argument of Furstenberg to prove that if a set $E \subseteq \mathbb{N}$ has positive density, then we can always find $x, y \in E$ such that $x - y$ is a perfect square.

(a) Let $f = \mathbb{1}_A$ and let μ_f be the spectral measure of f . Show that

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int \bar{f} \cdot T^{n^2} f d\mu = \mu_f(\{0\}) + \sum_{q \in \mathbb{Q} \cap [0,1]} \mu_f(\{q\}) S_q,$$

where

$$S_q = \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} e(qn^2).$$

Hint: Rewrite the left-hand side in terms of the spectral measure of f and use Weyl's theorem to eliminate the contribution of the irrationals in the integral.

Using the spectral theorem, we get a measure μ_f on the torus, such that

$$\int \bar{f} \cdot T^n f d\mu = \int e(nt) d\mu_f(t)$$

for every $n \in \mathbb{N}$. Therefore, the desired average is equal to

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int \bar{f} \cdot T^{n^2} f d\mu = \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N \int e(n^2 t) d\mu_f(t) = \lim_{N \rightarrow +\infty} \int \frac{1}{N} \sum_{n=1}^N e(n^2 t) d\mu_f(t)$$

When t is irrational, we have that

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N e(n^2 t) = 0,$$

since the sequence $(n^2t)_{n \in \mathbb{N}}$ is uniformly distributed (mod 1). On the other hand, in the case where t is rational, the sequence $e(n^2t)$ is periodic. Therefore, the averages converge to some number (denoted by S_q in the statement), which is bounded in modulus by 1. By the dominated convergence theorem, we conclude that

$$\lim_{N \rightarrow +\infty} \int \frac{1}{N} \sum_{n=1}^N e(n^2t) d\mu_f(t) = \int \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N e(n^2t) d\mu_f(t) = \int_{\mathbb{T}} \mathbf{1}_{x=q \in \mathbb{Q} \cap [0,1]} S_q d\mu_f(x).$$

We conclude that the original limit is equal to

$$\sum_{q \in \mathbb{Q} \cap [0,1]} \mu_f(\{q\}) S_q$$

and the conclusion follows by simply noting that $S_0 = 1$.

(b) Repeat the previous step to show that if W is any positive integer, then

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int \bar{f} \cdot T^{(Wn)^2} f d\mu = \sum_{q \in A_1} \mu_f(\{q\}) + \sum_{q \in (\mathbb{Q} \cap [0,1]) \setminus A_1} \mu_f(\{q\}) S_{W,q} \quad (1)$$

where A_1 consists of the rationals $q \in \mathbb{Q}$ for which W^2q is an integer and

$$S_{W,q} = \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} e(q(Wn)^2).$$

The proof is the same as in the previous part. We eventually find that the limit is equal to

$$\sum_{q \in (\mathbb{Q} \cap [0,1])} \mu_f(\{q\}) S_{W,q}$$

and we get the desired result by noting that if W^2q is an integer, then $e((W^2qn)^2) = 1$ for all $n \in \mathbb{N}$, which implies that $S_{W,q} = 1$.

(c) Show that we can pick W sufficiently large to make the contribution of the second sum negligible. Namely, for every $\varepsilon > 0$, show that we can pick $W \in \mathbb{N}$, such that

$$\lim_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} \int \bar{f} \cdot T^{(Wn)^2} f d\mu \right| \geq \mu_f(\{0\}) - \varepsilon.$$

Conclude that $\mu(A \cap T^{-(Wn)^2} A) > 0$ for infinitely many $n \in \mathbb{N}$.

Let $\varepsilon > 0$. We observe that the sum

$$\sum_{q \in \mathbb{Q} \cap [0,1]} \mu_f(\{q\})$$

converges. Therefore, there exists a finite set F of rationals, such that

$$\sum_{q \notin F} \mu_f(\{q\}) < \varepsilon.$$

We pick W to be the product of all the denominators appearing in the finite set F , so

that $Wq \in \mathbb{N}$ for every $q \in F$. Then, using part b) for this choice of W , we have that

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int \bar{f} \cdot T^{(Wn)^2} f \, d\mu = \sum_{q \in A_1} \mu_f(\{q\}) + \sum_{q \in (\mathbb{Q} \cap [0,1]) \setminus A_1} \mu_f(\{q\}) S_{W,q}.$$

Using the triangle inequality, we deduce that the left-hand side in the previous equality is larger than

$$\left| \sum_{q \in A_1} \mu_f(\{q\}) \right| - \left| \sum_{q \in (\mathbb{Q} \cap [0,1]) \setminus A_1} \mu_f(\{q\}) S_{W,q} \right| \geq \sum_{q \in A_1} \mu_f(\{q\}) - \sum_{q \in (\mathbb{Q} \cap [0,1]) \setminus A_1} \mu_f(\{q\}) |S_{W,q}| \geq \mu_f(\{0\}) - \sum_{q \in (\mathbb{Q} \cap [0,1]) \setminus A_1} \mu_f(\{q\}) > \mu_f(\{0\}) - \varepsilon,$$

since

$$\sum_{q \in (\mathbb{Q} \cap [0,1]) \setminus A_1} \mu_f(\{q\}) \leq \sum_{q \notin F} \mu_f(\{q\}) < \varepsilon.$$

We infer that

$$\lim_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} \int \bar{f} \cdot T^{(Wn)^2} f \, d\mu \right| \geq \mu_f(\{0\}) - \varepsilon$$

or, equivalently ($f = \mathbb{1}_A$),

$$\lim_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-(Wn)^2} A) \right| \geq \mu_f(\{0\}) - \varepsilon.$$

By Exercise 1, we have

$$\mu_f(\{0\}) = \|P_T \mathbb{1}_A\|_{L^2(\mu)}^2 \geq \|P_T \mathbb{1}_A\|_{L^1(\mu)}^2 = \left(\int |P_T \mathbb{1}_A| \, d\mu \right)^2 \geq \left(\int P_T \mathbb{1}_A \, d\mu \right)^2 = (\mu(A))^2,$$

since

$$\int P_T \mathbb{1}_A \, d\mu = \int \mathbb{1}_A \, d\mu + \int_{\in H_{\text{erg}}} (P_T \mathbb{1}_A - \mathbb{1}_A) \, d\mu = \int \mathbb{1}_A \, d\mu.$$

Picking ε sufficiently small, the theorem follows.